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Application of the Method of Averages to Celestial Mechanics

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Formulas are derived giving the second-order secular and long-period effects on the orbit of the satellite of an oblate planet. The presentation is in terms of the standard osculating elements and their averages.

The derivation involves the second-order method of averages for solving nonlinear differential equations. In the application to the present problem, this second-order theory can be very cumbersome. However, by introducing an auxiliary function, ψ, it has been possible to simplify the equations to a manageable form. Author

I. INTRODUCTION

The method of averages developed by N. M. Krylov and N. N. Bogoliubov in the study of nonlinear oscillations has been generalized by Bogoliubov and J. Mitropolsky in their treatise *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Ref. 1). For the treatment of oscillating systems, the methods developed by these authors have not as yet been fully exploited by astronomers. For example, using the averaging method in satellite theory the secular and long-period rates of the osculating elements can be expressed through second-order terms in the small parameter, with coefficients exact in eccentricity, thus avoiding expansions in powers of the eccentricity.

The labor involved in the derivations is lengthy but is believed to be considerably simpler than, for example, the von Zeipel method used by Brouwer, Kozai, and others (Refs. 2, 3). As far as can be ascertained, the only other attempt at applying the second-order averaging method to celestial mechanics is currently being pursued by W. T. Kyner (Ref. 4), whose approach deals directly with the equations of motion rather than with the canonical form used here. It is believed that a saving of labor is effected and that symmetries are better exhibited by the canonical approach. As of the present time, Kyner's second-order results (referred to in Ref. 4) have not been available to the authors of this paper.

To check the validity of the results obtained herein, a comparison is made with the formulas of Petty and Breakwell (Ref. 5). Numerical computations were also performed, but neither check was entirely satisfactory. These evaluations are discussed in detail in Sections VI and VII.

This paper consists of three parts. In Section II the averaging procedure is developed and formulas appropriate for the subsequent analysis are derived. The derivations here are simpler and more straightforward than those found in Ref. 1. In Sections III, IV, and V, the theory is applied to a satellite problem of celestial mechanics, with particular emphasis on the oblateness perturbations. Finally, in Sections VI and VII, the results are evaluated in terms of other methods, and numerical results are stated.

II. THE METHOD OF AVERAGES

We consider the system of nonlinear differential equations

$$\frac{dx^i}{dt} = \epsilon X^i(x,t) + \epsilon^2 Y^i(x,t) \qquad \epsilon \ll 1$$

$$i = 1, 2, \cdots, n \tag{1}$$

with $X^i(x,t) = X^i(x^1, x^2, \cdots, x^n, t)$, etc., ϵ a small parameter. The independent variable t is such that $X^i(x,t)$, $Y^i(x,t)$ are periodic in t, of period τ, so that $X^i(x,t+\tau) = X^i(x,t)$, $Y^i(x,t+\tau) = Y^i(x,t)$. We will be interested only in terms through ϵ^2 (second-order perturbation theory). The equations for perturbed Keplerian orbits can be cast in this form by use of the method of variation of parameters (Ref. 6). Alternatively, such motion may be described in terms of the Lagrange planetary equations which are already of this form (Ref. 3).

We wish to find $x^i(t)$, $i = 1, 2, \cdots, n$, which satisfy Eq. (1) to within ϵ^3 terms. The method of averages is basically a scheme for separating $x^i(t)$ into two parts, a steady-state or long-period term and a short-period term. The procedure, following Bogoliubov and Mitropolsky[1], is to introduce a smooth set of variables ξ^i, $i = 1, 2, \cdots, n$, which behave like x^i without the short period terms.

We define

$$\frac{d\xi^i}{dt} = \epsilon A^i(\xi) + \epsilon^2 B^i(\xi) \tag{2}$$

$$x^i = \xi^i + \epsilon F^i(\xi,t) + \epsilon^2 G^i(\xi,t) \tag{3}$$

in which the F^i, G^i are periodic in t, of period τ, with no steady-state components, and the A^i, B^i are independent of t. Thus

$$\int_0^\tau F^i(\xi,t)\,dt = \int_0^\tau G^i(\xi,t)\,dt = 0 \qquad i = 1, 2, \cdots, n \tag{4}$$

[1] See Ref. 1, Eqs. (24.63) and (24.64).

$$\frac{\partial A^i}{\partial t} = \frac{\partial B^i}{\partial t} = 0 \tag{5}$$

with the ξ^i held constant during the integration.

We wish to determine the A^i, B^i, F^i, G^i such that x^i of Eq. (3) will satisfy Eq. (1) to within ϵ^3 terms. One advantage of Eqs. (2) and (3) over Eq. (1) is that (for satellite lifetime studies) a knowledge of $\xi^i(t)$ yields the long-term behavior of $x^i(t)$, since $x^i(t) - \xi^1(t)$ merely defines the short-period fluctuations from the average motion. Furthermore, Eq. (3) can be integrated numerically using much larger integration steps than can Eq. (1).

From Eqs. (3) and (1) we have

$$\begin{aligned}\frac{dx^i}{dt} &= \frac{d\xi^i}{dt} + \epsilon \frac{\partial F^i}{\partial \xi^\alpha}\frac{d\xi^\alpha}{dt} + \epsilon \frac{\partial F^i}{\partial t} + \epsilon^2 \frac{\partial G^i}{\partial t} \\ &= \epsilon A^i(\xi) + \epsilon \frac{\partial F^i}{\partial t} + \epsilon^2 \left[A^\alpha(\xi)\frac{\partial F^i}{\partial \xi^\alpha} + \frac{\partial G^i}{\partial t} + B^i(\xi)\right] \\ &= \epsilon X^i(\xi,t) + \epsilon^2 \frac{\partial X^i}{\partial \xi^\alpha} F^\alpha(\xi,t) + \epsilon^2 Y^i(\xi,t)\end{aligned} \tag{6}$$

neglecting ϵ^3 and higher terms and making use of the summation convention. The right-hand side of Eq. (6) results from a Taylor series expansion about $x^i = \xi^i$.

Equating powers of ϵ yields

$$X^i(\xi,t) = A^i(\xi) + \frac{\partial F^i(\xi,t)}{\partial t}$$

$$Y^i(\xi,t) + F^\alpha(\xi,t)\frac{\partial X^i(\xi,t)}{\partial \xi^\alpha} = A^\alpha(\xi)\frac{\partial F^i(\xi,t)}{\partial \xi^\alpha} + \frac{\partial G^i(\xi,t)}{\partial t} + B^i(\xi) \tag{7}$$

From Eqs. (7) it follows that $A^i(\xi)$ is the steady-state term of $X^i(\xi,t)$, and $\partial F^i/\partial t$ is the periodic part of $X^i(\xi,t)$. Thus[2]

$$A^i(\xi) = \frac{1}{\tau}\int_0^\tau X^i\,(\xi,t)\,dt = X_0^i(\xi) \tag{8}$$

$$F^i(\xi,t) = \int^t X_p^i(\xi,t)\,dt = \tilde{X}^i \tag{9}$$

with $X^i(\xi,t) = X_0^i(\xi) + X_p^i(\xi,t)$, and the integral of Eq. (9) being the indefinite integral of $X_p^i(\xi,t)$ involving no constants of integrations, denoted by $\tilde{X}^i$.

From Eqs. (7) it also follows that

$$B^i(\xi) = \frac{1}{\tau}\int_0^\tau Y^i(\xi,t)\,dt + \frac{1}{\tau}\int_0^\tau \tilde{X}^\alpha \frac{\partial X^i}{\partial \xi^\alpha}\,dt \tag{10}$$

$$G^i(\xi,t) = \tilde{Y}^i + \int^t \left(\tilde{X}^\alpha \frac{\partial X^i}{\partial \xi^\alpha}\right)_p dt - X_0^\alpha(\xi)\int^t \frac{\partial \tilde{X}^i}{\partial \xi^\alpha}\,dt \tag{11}$$

Returning to Eqs. (2) and (3), one notes that

$$\frac{d\xi^i}{dt} = \epsilon\, X_0^i(\xi) + \epsilon^2\left[Y_0^i(\xi) + \left(\tilde{X}^\alpha \frac{\partial X^i}{\partial \xi^\alpha}\right)_0\right] \tag{12}$$

[2]The subscript 0 denotes steady-state terms, and the subscript p denotes the purely periodic part of the functions. In evaluating the integrals (8) and (9) it is assumed that ξ is held constant.

$$x^i = \xi^i + \epsilon \tilde{X}^i + \epsilon^2 \left[\tilde{Y}^i + \int^t \left(\tilde{X}^\alpha \frac{\partial X^i}{\partial \xi^\alpha} \right)_p dt - X_0^\alpha(\xi) \int^t \frac{\partial \tilde{X}^i}{\partial \xi^\alpha} dt \right] \tag{13}$$

Equations (12) and (13) are the basic forms for applying the second-order averaging process. In Section III the equations for satellite theory will be cast in appropriate form, and the various functions appearing in Eqs. (12) and (13) will be evaluated in terms of the orbital elements.

III. EQUATIONS FOR SATELLITE MOTION

The specific problem analyzed here concerns satellite motion in the presence of a force field produced by a slightly distorted mass spheroid. Such a force field has a potential which can be expanded in spherical harmonics starting with

$$U = -\frac{\mu}{r} - \frac{\mu J}{r^3}\left(\frac{1}{3} - \sin^2\beta\right) + \cdots \tag{14}$$

Taking the equatorial plane of the spheroid as the reference plane, the latitude is β and μ is the gravity constant. The theory will be developed here for the J-term only, neglecting all higher-order terms. However, the solution to the corresponding satellite motion will be carried to order J^2, in the sense of the second-order averaging method as explained in Section II above.

Following Brouwer (Ref. 2), Delaunay variables are introduced as follows:

$$L = (\mu a)^{1/2} \qquad l = \text{mean anomaly}$$

$$G = L(1 - e^2)^{1/2} \qquad g = \text{argument of pericenter}$$

$$H = G\cos I \qquad h = \text{longitude of ascending node}$$

In terms of these variables, the Hamiltonian is given by

$$F = \frac{\mu^2}{2L^2} + \frac{\mu^4 J}{3L^6}\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3} + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos(2g + 2f)\right]$$

$$= \frac{\mu^2}{2L^2} + \frac{J\mu^4}{3G^6} R(L,G,H,l,g) \tag{15}$$

where R is given explicitly by Eq. (35), and the equations of motion are

$$\frac{dL}{dt} = \frac{\partial F}{\partial l} \qquad \frac{dl}{dt} = -\frac{\partial F}{\partial L}$$

$$\frac{dG}{dt} = \frac{\partial F}{\partial g} \qquad \frac{dg}{dt} = -\frac{\partial F}{\partial G} \tag{16}$$

$$\frac{dH}{dt} = \frac{\partial F}{\partial h} \qquad \frac{dh}{dt} = -\frac{\partial F}{\partial H}$$

Here, a, e, and I are the usual osculating elements, and f is the true anomaly.

The use of canonical variables with the corresponding simplicity of the equations of motion as exhibited in Eqs. (16) is essential to the present development from a practical standpoint. For, without such simplicity, the functions appearing on the right-hand side of Eqs. (12) and (13) become excessively complex.

The potential (Eq. 14) leads to a Hamiltonian (Eq. 15) independent of h and t. This fact further simplifies the calculation. However, the procedure would still apply were the Hamiltonian to contain h and t, provided only that t entered as a slowly varying term. All the short-period variations must be the result of terms involving the mean anomaly l and its multiples.

Equations (16) should now be rewritten with l as independent variable (corresponding to the independent variable t of Eq. 1). This transformation is effected by dividing the dl/dt equation into each of the remaining equations of the set (16).

If

$$\epsilon = \frac{J\mu^2}{3} \tag{17}$$

then

$$\frac{dl}{dt} = \frac{\mu^2}{L^3}\left(1 - \epsilon \frac{L^3}{G^6} \frac{\partial R}{\partial L}\right) \tag{18}$$

and to order ϵ^2 the equations of motion reduce to

$$\frac{dL}{dl} = \epsilon \frac{L^3}{G^6} \frac{\partial R}{\partial l} + \epsilon^2 \frac{L^6}{G^{12}} \frac{\partial R}{\partial l} \frac{\partial R}{\partial L} \tag{19}$$

$$\frac{dG}{dl} = \epsilon \frac{L^3}{G^6} \frac{\partial R}{\partial g} + \epsilon^2 \frac{L^6}{G^{12}} \frac{\partial R}{\partial g} \frac{\partial R}{\partial L} \tag{20}$$

$$\frac{dH}{dl} = \epsilon \frac{L^3}{G^6} \frac{\partial R}{\partial h} + \epsilon^2 \frac{L^6}{G^{12}} \frac{\partial R}{\partial h} \frac{\partial R}{\partial L} \tag{21}$$

$$\frac{dg}{dl} = -\epsilon L^3 \frac{\partial}{\partial G}\left(\frac{R}{G^6}\right) - \epsilon^2 \frac{L^6}{G^6} \frac{\partial R}{\partial L} \frac{\partial}{\partial G}\left(\frac{R}{G^6}\right) \tag{22}$$

$$\frac{dh}{dl} = -\epsilon \frac{L^3}{G^6} \frac{\partial R}{\partial H} - \epsilon^2 \frac{L^2}{G^{12}} \frac{\partial R}{\partial H} \frac{\partial R}{\partial L} \tag{23}$$

In comparing with Eq. (1), the dependent variables $x^j = x^1, \cdots, x^5$ are identified with L, G, H, g, and h, respectively, and the independent variable is l.

If in dealing with some other potential, t appears explicitly in F, or if it is desired to relate the orbit to time, then it is necessary to return to Eq. (18). By introducing the slowly varying variable η

$$\eta = t - \frac{1}{\mu^2} \int_0^l L^3 \, dl \tag{24}$$

the value of t can be derived from the equation

$$\frac{d\eta}{dl} = \epsilon \frac{L^6}{\mu^2 G^6} \frac{\partial R}{\partial L} + \epsilon^2 \frac{L^9}{\mu^2 G^{12}} \left(\frac{\partial R}{\partial L}\right)^2 \tag{25}$$

Here, η is to be considered as a sixth dependent variable.

The final step in the formulation of the equations is the identification of the functions X^j and Y^j. Comparison of Eq. (1) with Eqs. (19) to (25) shows that

$$
\begin{aligned}
X^1 &= X^L = \frac{L^3}{G^6}\frac{\partial R}{\partial l} & Y^1 &= Y^L = \frac{L^6}{G^{12}}\frac{\partial R}{\partial l}\frac{\partial R}{\partial L} \\
X^2 &= X^G = \frac{L^3}{G^6}\frac{\partial R}{\partial g} & Y^2 &= Y^G = \frac{L^6}{G^{12}}\frac{\partial R}{\partial g}\frac{\partial R}{\partial L} \\
X^3 &= X^H = \frac{L^3}{G^6}\frac{\partial R}{\partial h} & Y^3 &= Y^H = \frac{L^6}{G^{12}}\frac{\partial R}{\partial h}\frac{\partial R}{\partial L} \\
X^4 &= X^g = -L^3\frac{\partial}{\partial G}\left(\frac{R}{G^6}\right) & Y^4 &= Y^g = -\frac{L^6}{G^6}\frac{\partial R}{\partial L}\frac{\partial}{\partial G}\left(\frac{R}{G^6}\right) \\
X^5 &= X^h = -\frac{L^3}{G^6}\frac{\partial R}{\partial H} & Y^5 &= Y^h = -\frac{L^6}{G^{12}}\frac{\partial R}{\partial H}\frac{\partial R}{\partial L} \\
X^6 &= X^\eta = \frac{L^6}{\mu^2 G^6}\frac{\partial R}{\partial L} & Y^6 &= Y^\eta = \frac{L^9}{\mu^2 G^{12}}\left(\frac{\partial R}{\partial L}\right)^2
\end{aligned}
\tag{26}
$$

Section IV will be devoted to simplifying the form of Eqs. (12) and (13) after making the substitutions indicated by Eqs. (26).

IV. EQUATIONS FOR AVERAGED DELAUNAY VARIABLES

The variables will be considered one at a time. For L, the equation corresponding to Eq. (12) for $j = 1$ may be written[3]

$$\frac{d\bar{L}}{dl} = \epsilon X_0^L + \epsilon^2 \left[Y_0^L + \left(\tilde{X}^i \frac{\partial X_p^L}{\partial \bar{x}^i} \right)_0 \right] \tag{27}$$

Using the identification from Eq. (26), it is seen that

$$X_0^L = \frac{L^3}{G^6} \frac{1}{\tau} \int_0^\tau \frac{\partial R}{\partial l} \, dl = 0 \tag{28}$$

since the integration is effected holding the variables L, G, H, g, and h constant, and since R is assumed to be periodic in l.

The terms in brackets reduce to

$$L^6 \left(\phi_l \phi_L + \frac{3}{L} \phi \phi_l + \phi \phi_{lL} + \tilde{\phi}_g \phi_{Gl} + \tilde{\phi}_h \phi_{Hl} - \tilde{\phi}_G \phi_{gl} - \tilde{\phi}_H \phi_{hl} \right)_0 + \frac{L^9}{\mu^2} (\tilde{\phi}_L \phi_{\eta l})_0 \tag{29}$$

where

$$\phi = \frac{R}{G^6} \tag{30}$$

and subscripts denote partial derivatives.

[3] See footnote 2. Also, repeated index denotes use of summation convention.

In Eq. (29), the terms combine with the help of the following relations:

$$(\phi_l \phi_L)_0 = \frac{1}{\tau} \int_0^\tau \phi_l \phi_L \, dl = \frac{1}{\tau} (\phi \phi_L) \Big|_0^\tau - \frac{1}{\tau} \int_0^\tau \phi \phi_{lL} \, dl = -(\phi \phi_{lL})_0 \tag{31}$$

showing that the first and third terms cancel. Similarly the fourth and sixth terms cancel and the fifth and seventh terms cancel, leaving only the second and eighth terms. However,

$$(\phi \phi_l)_0 = \left(\frac{1}{2} \frac{\partial}{\partial l} \phi^2 \right)_0 = \frac{1}{2\tau} \phi^2 \Big|_0^\tau = 0 \tag{32}$$

Thus only one term remains in Eq. (27), which reduces to

$$\frac{d\bar{L}}{dl} = -\frac{\epsilon^2 L^9}{\mu^2} (\phi_L \phi_\eta)_0 = -\frac{\epsilon^2 L^9}{\mu^2 G^{12}} \left(\frac{\partial R_p}{\partial L} \frac{\partial R_p}{\partial \eta} \right)_0 \tag{33}$$

It follows from Eq. (33) that

$$\frac{d\bar{L}}{dl} = 0 \tag{34}$$

if time (hence also η) does not appear explicitly in R. Since this is the case for the perturbing function due to each of the zonal harmonics of an oblate planet, it follows that none of these gives rise to secular or long-period second-order terms in the semimajor axis.

The remaining part of the analysis will be restricted to the case in which R is due to the second zonal harmonic for an oblate spheroid. Thus R is determined by Eq. (15), and can be written

$$R = (1 + e \cos f)^3 \left[A + B \cos (2g + 2f) \right] \tag{35}$$

where

$$\left.\begin{aligned} A &= -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2} = 1 - B = 1 - \frac{3}{2}\sin^2 I \\ B &= \frac{3}{2}\left(1 - \frac{H^2}{G^2}\right) = 1 - A = \frac{3}{2}\sin^2 I \end{aligned}\right\} \tag{36}$$

Note that R is independent of h and that time does not appear explicitly. Also,[4]

$$R_0 = \frac{1}{2\pi}\int_0^{2\pi} R\,dl = \frac{1}{2\pi}\cdot\frac{G^3}{L^3}\int_0^{2\pi}(1 + e\cos f)^{-2} R\,df = A\frac{G^3}{L^3} \tag{37}$$

The equation for determining $\bar{G}$ is

$$\frac{d\bar{G}}{dl} = \epsilon X_0^G + \epsilon^2\left[Y_0^G + \left(\tilde{X}^i \frac{\partial X_p^G}{\partial \bar{x}^i}\right)_0\right] \tag{38}$$

The value of X_0^G is 0 since

$$X_0^G = \frac{L^3}{G^6}\frac{\partial}{\partial g}[R]_0 = \frac{L^3}{G^6}\frac{\partial}{\partial g}\left(\frac{AG^3}{L^3}\right) = 0 \tag{39}$$

Using Eq. (26), the terms in brackets of Eq. (38) reduce to

$$\begin{aligned} L^6 &\left(\phi_g\phi_L + \frac{3}{L}\phi\phi_g + \phi\phi_{gL} + \tilde{\phi}_g\phi_{gG} + \tilde{\phi}_h\phi_{gH} - \tilde{\phi}_G\phi_{gg} - \tilde{\phi}_H\phi_{gh}\right)_0 \\ &= L^6\frac{\partial}{\partial g}\left[\frac{1}{L^3G^{12}}\frac{\partial}{\partial L}\left(\frac{1}{2}L^3R^2\right)_0 + (\tilde{\phi}_g\phi_G)_0 + (\tilde{\phi}_h\phi_H)_0\right] \end{aligned} \tag{40}$$

[4] Although the notation requires that averaged variables be identified with a bar, this will not always be done in the sequel when the meaning is otherwise clear.

in which ϕ is given by Eq. (30). Since ϕ is independent of h, the last term in Eq. (40) can be omitted.

Write

$$\psi(L, G, H, g) = \frac{1}{G^{12}L^3} \frac{\partial}{\partial L} \left[L^3 \left(\frac{1}{2} R^2 \right)_0 \right] + (\tilde{\phi}_g \phi_G)_0 \tag{41}$$

then Eq. (38) reduces to

$$\frac{d\bar{G}}{dl} = \epsilon^2 L^6 \frac{\partial \psi}{\partial g} \tag{42}$$

Proceeding to the next variable, $\bar{H}$, it follows that

$$\frac{d\bar{H}}{dl} = 0 \tag{43}$$

since R is independent of h.

For $\bar{g}$ the situation is more complicated, since the term in ϵ is not zero. The rate equation is

$$\frac{d\bar{g}}{dl} = \epsilon X_0^g + \epsilon^2 \left[Y_0^g + \left(\tilde{X}^i \frac{\partial X_p^g}{\partial \bar{x}^i} \right)_0 \right] \tag{44}$$

with

$$X_0^g = -L^3 \frac{\partial}{\partial G} \left[\frac{1}{G^6} (R)_0 \right] = \frac{3A}{G^4} - \frac{1}{G^3} \frac{\partial A}{\partial G} = \frac{3}{G^4} \left(2 - \frac{5}{2} \sin^2 I \right) \tag{45}$$

while the terms of Eq. (44) in brackets reduce to

$$-L^6 \frac{\partial \psi}{\partial G}$$

Thus, Eq. (44) can be written

$$\frac{d\bar{g}}{dl} = \frac{3}{G^4}\left(2 - \frac{5}{2}\sin^2 I\right)\epsilon - L^6 \psi_G \epsilon^2 \tag{46}$$

Finally, the h equation is

$$\frac{d\bar{h}}{dl} = -\frac{3H}{G^5}\epsilon - L^6 \psi_H \epsilon^2 \tag{47}$$

It may be noted at this point that the equations for the averaged variable may be very simply expressed in terms of the function

$$\Gamma = \epsilon L^3 \phi_0 + \epsilon^2 L^6 \left[\tilde{\phi}_g \phi_G + \tilde{\phi}_h \phi_H + \frac{1}{L^3}\frac{\partial}{\partial L}\left(\frac{L^3 \phi^2}{2}\right)\right]_0$$

Thus, the averaged Delaunay variables satisfy the system of differential equations

$$\frac{d\bar{L}}{dl} = \Gamma_l$$

$$\frac{d\bar{G}}{dl} = \Gamma_g \qquad \frac{d\bar{g}}{dl} = -\Gamma_G$$

$$\frac{d\bar{H}}{dl} = \Gamma_h \qquad \frac{d\bar{h}}{dl} = -\Gamma_H$$

It remains to evaluate ψ_g, ψ_G, and ψ_H. The first term in ψ (Eq. 41) involves the average of $1/2\ R^2$, as follows:

$$
\begin{aligned}
\left[\frac{1}{2} R^2\right]_0 &= \frac{1}{2\pi} \cdot \frac{1}{2} \int_0^{2\pi} (1 + e \cos f)^6 \left[A + B \cos (2g + 2f)\right]^2 dl \\
&= \frac{G^3}{2L^3} \cdot \frac{1}{2\pi} \int_0^{2\pi} (1 + e \cos f)^4 \left[A + B \cos (2g + 2f)\right]^2 df \\
&= \frac{G^3}{2L^3} \left[\left(A^2 + \frac{1}{2} B^2\right)\left(1 + 3e^2 + \frac{3}{8} e^4\right)\right. \\
&= \left. A B e^2 \left(3 + \frac{1}{2} e^2\right) \cos 2g + \frac{B^2 e^4}{32} \cos 4g \right]
\end{aligned}
\tag{48}
$$

The second term of ψ in Eq. (41) is

$$
\left(\widetilde{\phi_g \phi_G}\right)_0 = \frac{1}{G^{12}} \left(\widetilde{R_g R_G}\right)_0 - \frac{6}{G^{13}} \left(\widetilde{R_g R}\right)_0 \tag{49}
$$

But

$$
\begin{aligned}
\left(\widetilde{R_g R_G}\right)_0 = \frac{B G^5}{2L^6} \left\{\left(3 + \frac{1}{3} B\right) + e^2 \left(2 + \frac{4}{3} B\right) - \left[6A + \left(1 - \frac{3}{2} A\right) e^2\right] \cos 2g \right. \\
\left. - \frac{B e^2}{8} \cos 4g \right\}
\end{aligned}
\tag{50}
$$

and

$$
\left(\widetilde{R_g R}\right)_0 = \frac{B G^6}{L^6} \left[\frac{1}{2} B + e^2 \left(\frac{B}{3} + \frac{A}{2} \cos 2g\right)\right] \tag{51}
$$

Thus

$$\psi = \left[\left(3 - \frac{9}{2} B + \frac{5}{3} B^2\right) + \left(\frac{3}{4} - \frac{1}{2} B - \frac{5}{24} B^2\right) e^2 + \frac{5B - 7}{4} Be^2 \cos 2g\right] \cdot \frac{1}{L^6 G^7} \tag{52}$$

The derivatives of ψ are then

$$\frac{\partial\psi}{\partial g} = -\frac{5B - 7}{2} \frac{Be^2}{L^6 G^7} \sin 2g \tag{53}$$

$$\frac{\partial\psi}{\partial G} = \frac{1}{L^6 G^8} \left\{ 6 - \frac{23}{2} B + \frac{55}{12} B^2 + \frac{e^2}{4} (21 - 9B + 25B^2) \right.$$
$$\left. + \cos 2g \left[\frac{B}{2} (7 - 5B) + e^2 \left(\frac{21}{4} + \frac{3}{2} B - \frac{15}{4} B^2\right)\right] \right\} \tag{54}$$

$$\frac{\partial\psi}{\partial H} = \frac{3 \cos I}{L^6 G^8} \left[\left(\frac{9}{2} - \frac{5}{3} B\right) + \left(\frac{1}{2} + \frac{5}{12} B\right) e^2 + \left(\frac{7}{4} - \frac{5}{2} B\right) e^2 \cos 2g\right] \tag{55}$$

and the rate equations are (with $B = 3/2 \sin^2 I$):

$$\frac{d\overline{L}}{dl} = 0 \tag{56}$$

$$\frac{1}{\overline{G}} \frac{d\overline{G}}{dl} = \frac{5}{8} \frac{J^2 e^2}{p^4} \sin^2 I \sin 2g \left(\frac{14}{15} - \sin^2 I\right) \tag{57}$$

$$\frac{d\overline{H}}{dl} = 0 \tag{58}$$

$$\frac{d\bar{g}}{dl} = \frac{J}{p^2}\left(2 - \frac{5}{2}\sin^2 I\right) - \frac{J^2}{p^4}\left\{2 - \frac{23}{6}B + \frac{55}{36}B^2 \right.$$

$$\left. + \frac{e^2}{4}\left(7 - 3B + \frac{25}{3}B^2\right) + \cos 2g\left[\frac{B}{6}(7 - 5B) + e^2\left(\frac{7}{4} + \frac{B}{2} - \frac{5}{4}B^2\right)\right]\right\} \tag{59}$$

$$\frac{d\bar{h}}{dl} = -\frac{J}{p^2}\cos I - \frac{J^2\cos I}{p^4}\left[\frac{3}{2} - \frac{5}{9}B + \left(\frac{1}{6} + \frac{5}{36}B\right)e^2 + \left(\frac{7}{12} - \frac{5}{6}B\right)e^2\cos 2g\right] \tag{60}$$

V. EQUATIONS FOR THE OSCULATING ELEMENTS a, e, I, ω, Ω

Since the osculating elements are simply related to the Delaunay variables (see Section III), it is merely a matter of direct substitution to obtain the corresponding rate equations.

Before writing these, however, it is pertinent to note that, for second-order rates, the substitution procedure is valid. Consider, for example, a function $y = f(x)$ of a slowly varying variable x with periodic part x_p of the form

$$x = \bar{x} + \epsilon x_p + \epsilon^2 u_p$$

$$\dot{\bar{x}} = \epsilon \phi(\bar{x})$$

Then

$$y = f(\bar{x}) + f'(\bar{x})(\epsilon x_p + \epsilon^2 u_p) + \frac{1}{2} f''(\bar{x}) \epsilon^2 x_p^2 + \cdots$$

and

$$\bar{y} = f(\bar{x}) + \frac{1}{2} \epsilon^2 f''(\bar{x}) (x_p^2)_0$$

in which $[x_p^2]_0$ is the average of x_p^2 over one period and is a function of $\bar{x}$. It follows then that

$$\begin{aligned}\dot{\bar{y}} &= f'(\bar{x}) \dot{\bar{x}} + \frac{1}{2} \epsilon^2 [f''(\bar{x}) (x_p^2)_0] \dot{\bar{x}} + \cdots \\ &= f'(\bar{x}) \dot{\bar{x}} + g(\bar{x}) \epsilon^3 + \cdots \qquad (61) \\ &= f'(\bar{x}) \dot{\bar{x}}\end{aligned}$$

through second-order terms in ϵ provided $g(\bar{x})$ is not of order $1/\epsilon$.

Assuming, therefore, that direct substitution is permissible, the desired second-order rate equations are

$$\frac{d\bar{a}}{dl} = 0 \tag{62}$$

$$\frac{d\bar{e}}{dl} = -\frac{5}{8}\frac{J^2}{p^4} e\,(1 - e^2)\sin^2 I \sin 2\omega \left(\frac{14}{15} - \sin^2 I\right) \tag{63}$$

$$\frac{d\bar{I}}{dl} = \frac{5}{16}\frac{J^2}{p^4} e^2 \sin 2I \sin 2\omega \left(\frac{14}{15} - \sin^2 I\right) \tag{64}$$

$$\begin{aligned}\frac{d\bar{\omega}}{dl} = \frac{J}{p^2}\left(2 - \frac{5}{2}\sin^2 I\right) - \frac{J^2}{p^4}\Bigg\{ & 2 - \frac{23}{4}\sin^2 I \\ & + \frac{55}{16}\sin^4 I + \frac{e^2}{4}\left(7 - \frac{9}{2}\sin^2 I + \frac{75}{4}\sin^4 I\right) \\ & + \frac{\cos 2\omega}{4}\left[\left(7 - \frac{15}{2}\sin^2 I\right)\sin^2 I + e^2\left(7 + 5\sin^2 I - \frac{45}{4}\sin^4 I\right)\right]\Bigg\}\end{aligned} \tag{65}$$

$$\frac{d\bar{\Omega}}{dl} = -\frac{J}{p^2}\cos I - \frac{J^2\cos I}{p^4}\left[\frac{3}{2} - \frac{5}{6}\sin^2 I + \left(\frac{1}{6} + \frac{5}{24}\sin^2 I\right)e^2 + \left(\frac{7}{12} - \frac{5}{4}\sin^2 I\right)e^2\cos 2\omega\right] \tag{66}$$

VI. COMPARISON WITH PREVIOUSLY PUBLISHED RESULTS

In (5), Petty and Breakwell develop formulas comparable to those listed in Eqs. (62) to (66). The referenced formulas, however, refer to variables associated with a reference plane somewhat different from the osculating plane, and, furthermore, the derivatives are given with respect to true anomaly rather than mean anomaly. In spite of these differences, the resulting rate formulas coincide very closely with the present results.

In fact, comparing the two sets of equations term by term shows that dI/dl agrees exactly; $d\Omega/dl$ agrees except for one term, namely the $5/6 \sin^2 I$ term in (66), which is given as $13/6 \sin^2 I$ in Ref. 5; $d\omega/dl$ does not agree except in form; de/dl does not agree.

Whether the two methods should agree to second order in J is not clear. However, the present formulas have been reviewed very thoroughly, and hopefully have no errors due to incorrect manipulation.

Actually, the de/dl and dI/dl equations are easily checked against each other, since

$$\frac{d\bar{e}}{dl} = -\frac{G}{eL^2}\frac{d\bar{G}}{dl}$$

and

$$\frac{d\bar{I}}{dl} = \frac{H}{G^2 \sin I}\frac{d\bar{G}}{dl}$$

Thus

$$\frac{d\bar{e}}{dl} = -\frac{1-e^2}{e}\frac{\sin I}{\cos I}\frac{d\bar{I}}{dl}$$

which checks for the present results, but not for Ref. 5.

VII. COMPARISON WITH NUMERICAL CHECK

To compute the averaged orbital elements to the accuracy required for checking Eqs. (62) to (66) to order J^2 is a touchy business at best. In any numerical example, the integration errors due to truncation and roundoff limit the precision of the computation. On the other hand, the scaling is constrained by the size of the first-order short-period fluctuations, so that it does not help matters to magnify the perturbing forces beyond a certain amount.

Therefore, only a token numerical check was attempted. Using a 7094 computer, the equations of motion for an Earth satellite perturbed only by the J-term were integrated for 30 days (about 360 periods). The value of J was taken as 0.00162345. The orbital semimajor axis was 7985 km, giving a period of 1 hr 58 min.

To obtain the averages of inclination and eccentricity, these quantities were first graphed for one complete period at the start and one complete period at the end of the orbit as shown in Figs. 1 and 2. Then, using points read off the graphs[5], the averages were computed by Simpson's rule. The results are shown in the table:

	0-day	30-day	Differences $\times 10^4$	
			Computed	Theoretical
$\bar{e}$	0.12314428	0.12310912	−0.3516	−0.2463
$\bar{I}$, deg	54.083031	54.083434	+4.03	+1.28

In reading the graphs, the fifth decimal place in eccentricity and the fourth place in inclination are questionable. Thus, the differences are questionable even in the first digit indicated. It can be concluded, only therefore, that the computation is compatible with the theory — a result which is all that could be hoped for without a great deal more effort. The theoretical values of the differences shown above are obtained by the method described in the Appendix.

[5] For the actual computation the graphs were scaled several times larger than those shown.

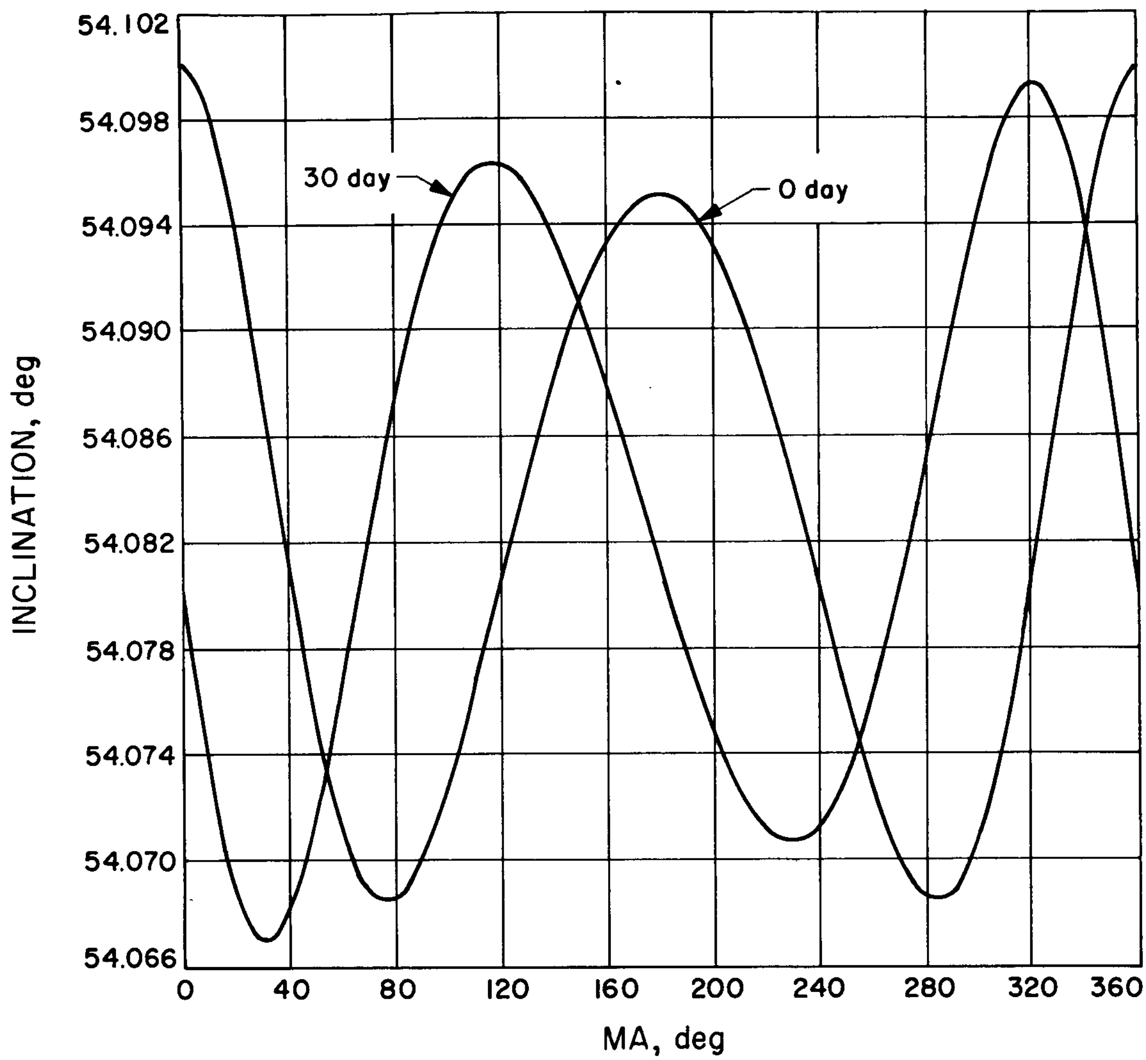

Fig. 1. Earth-satellite orbit: variation of osculating inclination over one orbital period, initially and after 30 days

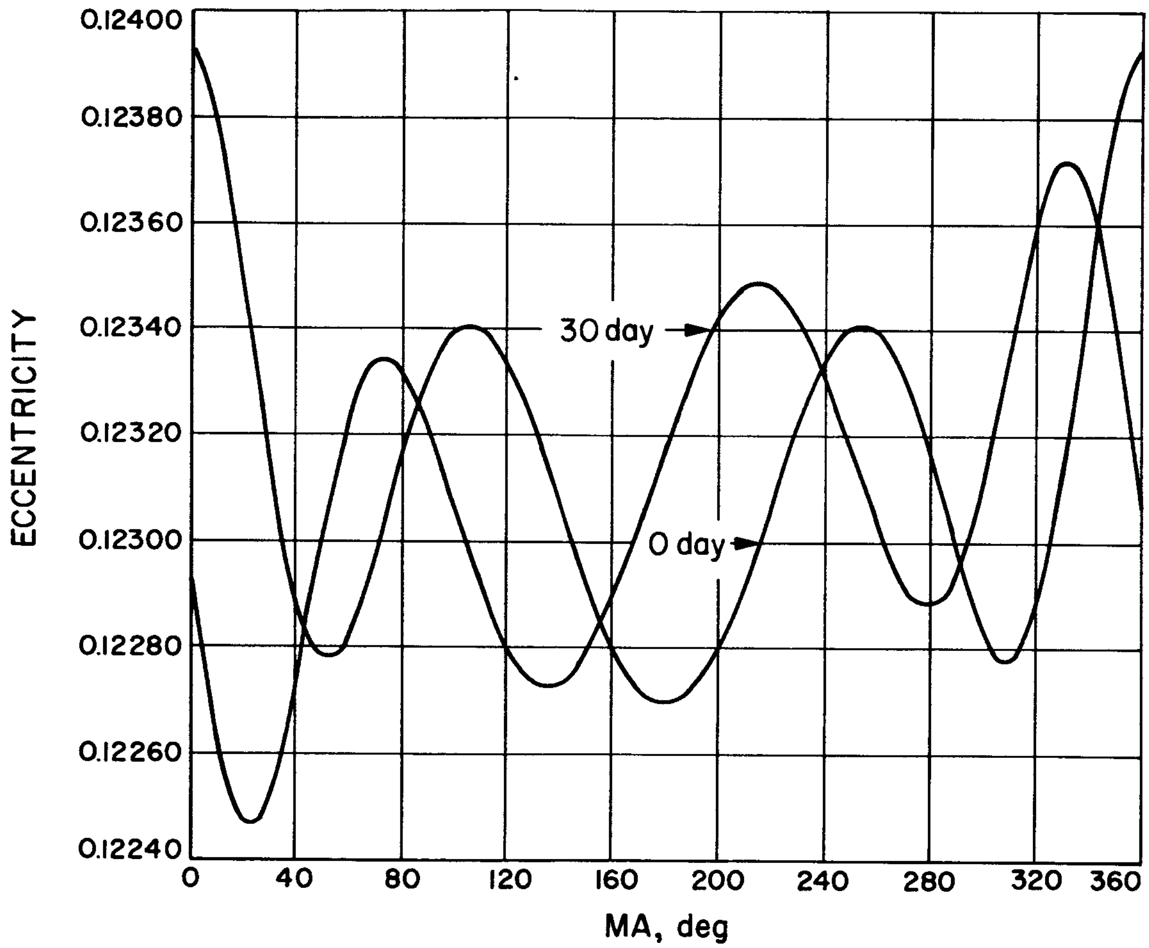

Fig. 2. Earth-satellite orbit: variation of osculating eccentricity over one orbital period, initially and after 30 days

VIII. CONCLUSIONS

The second-order long-period and secular terms in the rates of the osculating elements of the orbit of the satellite of an oblate planet have been computed by the method of averages. The derived formulas are valid for all eccentricity and all inclination. Furthermore, the differential equation is satisfied to second order uniformly in time.

The results are applicable to the study of satellite lifetimes for which short-period terms are not pertinent. In particular, by combining these formulas with first-order rate terms in the higher-order harmonics and third-body terms, it should be possible to construct an accurate lifetime computer program. In such a program for a lunar satellite, for example, it would be possible to use an integration step size of the order of a day.

Expansion of the present results to include second-order short-period terms is a matter of some effort but is certainly feasible. The method could also be applied to other harmonics and to third-body perturbations. On the other hand, application to terms of third order and higher, though possible, appears to be too involved to be worth while.

One of the strong points of the method of averages, as opposed, say, to the von Zeipel method, is that the results are represented in terms of familiar variables. There is no succession of transformations or use of auxiliary planes or other devices to distract the user from the basic quantities in which he is interested.

No attempt has been made in the present paper to integrate the rate equations or to discuss the behavior of the solution. Sufficient analysis of this type has already appeared in the literature, e.g., Ref. 5. Furthermore, in any application, there is bound to be a combination of many perturbing factors, so that an analysis which includes only the J and J^2 equations would not be valid.

NOMENCLATURE

A	see Eq. (36)
A^i	auxiliary function definitions implicit in Eq. (3)
B	see Eq. (36)
B^i	auxiliary function definition implicit in Eq. (3)
e	eccentricity of osculating conic
$\bar{e}$	$(1 - \bar{G}^2/\bar{L}^2)^{1/2}$
f	true anomaly
$F^i = (F^1, \cdots, F^n)$	auxiliary function definitions implicit in Eq. (2)
F	Hamiltonian (see Eq. 15)
g	argument of pericenter
$\bar{g}$	smoothed value of g
$G^i = (G^1, \cdots, G^n)$	auxiliary function definition implicit in Eq. (2)
G	$L(1 - e^2)^{1/2}$
$\bar{G}$	smoothed value of G
h	longitude of ascending node of osculating ellipse
$\bar{h}$	smoothed value of h
H	$G \cos I$
$\bar{H}$	smoothed value of H
I	inclination of plane of osculating orbit to equator
$\bar{I}$	smoothed value of I
J	coefficient of second sectoral harmonic in expansion of gravity field (see Section III)
l	mean anomaly
L	$\mu^{1/2}\, \alpha^{1/2}$
$\bar{L}$	smoothed value of L

NOMENCLATURE (Cont'd)

p	semilatus rectum
r	radial distance from center of mass
R	perturbing part of potential (see Eq. 15)
R_0	nonperiodic part of R
t	time (or independent variable)
U	potential function (see Section III)
$x = (x^1, \cdots, x^n)$	n dependent variables (see Eq. 2)
$\bar{x}^i$	smoothed value of x^i(see Eq. 3)
$X = (X^1, \cdots, X^n)$	functions in d. e. (Eq. 2)
X^j_0	nonperiodic part of X^j
X^j_p	periodic part of X^j
$\tilde{X}^j$	integral of X^j_p (see Eq. 7)
X^g, X^L, X^G	functions defined in Eq. (26)
$Y = (Y^1, \cdots, Y^n)$	function in d. e. (Eq. 2)
Y^j_0	nonperiodic part of Y^j
Y^j_p	periodic part of Y^j
$\tilde{Y}^j$	integral of Y^j_p (see Eq. 7)
Y^g, Y^L, Y^G	function defined in Eq. (26)
α	semimajor axis of osculating ellipse
μ	gravity constant (GM)
ξ	small parameter (see Eq. 17)
τ	period of periodic variable
ϕ	R/G^6 defined in Eq. (30)
ψ	defined in Eq. (41)

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APPENDIX. Integration of the Rate Equations

For purposes of numerical check, it is convenient to obtain an analytic integral of the rate equations (Eqs. 61 to 65). This is effected by the usual iterative procedure. A first-order solution of the equations, which is easily obtained, is substituted back into the equations, which then can be integrated to yield a second approximation.

To illustrate the procedure, consider the rate equation for $\bar{e}$ (Eq. 62). Since, in the first-order solution of the complete system, only $\bar{\omega}$ and $\bar{\Omega}$ are not constant, it is sufficient to consider $\bar{\omega}$ as the only variable in the right-hand side of Eq. (63), with

$$\bar{\omega} = \frac{J}{p^2}\left(2 - \frac{5}{2}\sin^2 I\right) t \tag{A-1}$$

Substituting in Eq. (63) and integrating yields

$$\frac{d\bar{e}}{d\omega} = -\frac{5}{8}\,\frac{J}{p^2}\,\frac{e(1-e^2)}{\left(2 - \frac{5}{2}\sin^2 I\right)}\left(\frac{14}{15} - \sin^2 I\right)\sin^2 I \sin 2\omega \tag{A-2}$$

and

$$\bar{e} - \bar{e}_0 = \frac{5}{16}\,\frac{J}{p^2}\,\frac{e(1-e^2)}{\left(2 - \frac{5}{2}\sin^2 I\right)}\left(\frac{14}{15} - \sin^2 I\right)\sin^2 I(\cos 2\omega - \cos 2\omega_0) \tag{A-3}$$

This last equation serves as the basis for the theoretical value given in the numerical check. The particular example in the text involves a net change in ω of approximately 50.90 deg.

The corresponding formula for the change (in degrees) of inclination is

$$I - I_0 = -\left(\frac{e \cot I}{1 - e^2} \cdot \frac{180}{\pi}\right)(e - e_0) \tag{A-4}$$